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Asymptotic Behavior of a Stationary Silo with Absorbing Walls

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We study the nearest neighbors one dimensional uniform q-model of force fluctuations in bead packs,(1) a stochastic model to simulate the stress of granular media in two dimensional silos. The vertical coordinate plays the role of time, and the horizontal coordinate the role of space. The process is a discrete time Markov process with state space $\mathbb{R}^{\{1,\dots,N\}}$. At each layer (time), the weight supported by each grain is a random variable of mean one (its own weight) plus the sum of random fractions of the weights supported by the nearest neighboring grains at the previous layer. The fraction of the weight given to the right neighbor of the successive layer is a uniform random variable in $[0,1]$ independent of everything. The remaining weight is given to the left neighbor. In the boundaries, a uniform fraction of the weight leans on the wall of the silo. This corresponds to *absorbing boundary conditions*. For this model we show that there exists a unique invariant measure. The mean weight at site i under the invariant measure is $i(N+1-i)$; we prove that its variance is $\frac{1}{2}(i(N+1-i))^2+O(N^3)$ and the covariances between grains $i\neq j$ are of order $O(N^3)$. Moreover, as $N\to\infty$, the law under the invariant measure of the weights divided by N^2 around site (integer part of) rN, $r\in(0,1)$, converges to a product of gamma distributions with parameters 2 and $2(r(1-r))^{-1}$ (sum of two exponentials of mean $r(1-r)/2$). Liu *et al.*(2) proved that for a silo with infinitely many weightless grains, any product of gamma distributions with parameters 2 and $2/\rho$ with $\rho\in[0,\infty)$ are invariant. Our result shows that as the silo grows, the model selects exactly one of these Gamma's at each macroscopic place.

KEY WORDS: Granular media; silos stress.

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1. INTRODUCTION

In the nearest neighbors one dimensional uniform q-model of force fluctuations in bead packs the grains are arranged in layers inside a silo; each grain has random weight with mean 1 and supports part of the weight of the grains of higher layers. The weight supported by each grain (plus its own weight) is randomly distributed between the two neighboring grains of the following lower layer. The fraction of the weight given to the right nearest neighbor is chosen uniformly in the interval [0, 1]; the left nearest neighbor receives the complementary weight. The walls of the silo absorb a random uniform fraction of the weight supported by the boundary grains. The resulting model is described by the following evolution equations. Fix a silo width N and for $i \in \{1,\dots, N\}$:

$$W_t^N(i) = V_t(i) + W_{t-1}^N(i+1)\, U_{t-1}(i+1) + W_{t-1}^N(i-1)(1-U_{t-1}(i-1)), \tag{1.1}$$

$$W_t^N(0) = W_t^N(N+1) \equiv 0, \tag{1.2}$$

where $V_t(i)$ is the random weight of grain i of level t and $U_t(i)$ is the random fraction of the weight grain i of layer t gives to grain $i-1$ of layer $t+1$. The boundary conditions (1.2) represent the absorbing wall. The random weights $V_t(i)$ have mean one: $\mathbb{E}V_t(i) = 1$. The process W_t^N is Markov in the state space $(\mathbb{R}^+)^N$. We are interested in stationary measures μ^N for this process. Under a stationary measure, the mean weights $w^N(i)$ must satisfy the following equations. For $i \in \{1,\dots, N\}$:

$$\begin{aligned} w^N(i) &= 1 + \tfrac{1}{2} w^N(i-1) + \tfrac{1}{2} w^N(i+1), \\ w^N(0) &= w^N(N+1) = 0. \end{aligned} \tag{1.3}$$

obtained by taking expectations in (1.1) and assuming that the law at time t is the same as the law at time $t-1$. These equations have a unique solution, that turns out to be quadratic:

$$w^N(i) = i(N+1-i). \tag{1.4}$$

This implies that all candidates to be stationary measures must have the same mean profile (1.4). Our first result uses (1.4) to show the existence of an invariant measure. Indeed, starting with any measure with this quadratic profile, the means will be the same at all times. This guarantees that the sequence of measures indexed by times $t \in \{0, 1, 2, \ldots\}$ is tight (i.e., the mass does not escape to infinity; see the formal definition after display (3.2) below) and by taking Cesàro limits along convergent subsequences, one can show that there exists a stationary measure. To prove uniqueness we realize simultaneously two versions of the process starting with different initial invariant distributions using the same sequence U_t for both evolutions. This is called *coupling* in the probabilistic literature. Under the coupling all weight added to the system after time zero evolves identically in both versions, while the discrepant weight present at time zero is distributed among the neighbors to eventually get lost at the boundaries. As a consequence, both versions will have the same distribution in the limit as $t \to \infty$. This is the key ingredient to show that there is a *unique* invariant measure μ^N for the silo.

Next step is to study properties of the invariant measure μ^N. It seems hard to explicitly describe the invariant measure for each finite N, but it is possible to study the asymptotic behavior when N grows. We show that the covariances between the weights supported by different grains under the invariant measure are bounded by a constant times N^3 and that the variance at site i differs from $(w^N(i))^2/2$ by at most a constant times N^3. This implies that if one normalizes the stationary weights W^N (distributed according to μ^N) dividing by N^2, one obtains, in the limit when $N \to \infty$, that the covariances of the normalized weights vanish and that the variance of $W^N([rN])/N^2$ converges to $(r(1-r))^2/2$. Here $[rN]$ is the integer part of rN.

We also show that the law of the vector W^N/N^2 around site $[rN]$ converges to a product of gamma distributions with parameters 2 and $2/(r(1-r))$, the sum of two independent exponentials of mean $r(1-r)/2$ each. To understand this result divide (1.1) by N^2 and take N to infinity to get that the limiting values must satisfy the same equation but in the infinite lattice $\mathbb{Z}$ and without the V_t (weightless grains). The limiting equations correspond to the evolution of a model that we call *infinite silo model* (ISM). As Coppersmith *et al.*[1] did for the periodic case, a computation shows that any space-homogeneous product of gamma distributions is invariant for the ISM. The key observation to show this is the fact that if X and Y are independent exponentials with mean ρ and U is independent of X, Y and uniformly distributed in $[0, 1]$, then $U(X+Y)$ and $(1-U)(X+Y)$ are again independent exponentials with mean ρ. Actually we show more: following an approach of Liggett[3] we prove that *all* invariant measures for the ISM that are also shift invariant are mixtures of products of gammas with parameters 2 and ρ, with $\rho \in \mathbb{R}^+$. Hence, the distribution of the vector $(W^N([rN]+\ell)/N^2 : \ell \in \mathbb{Z})$ converges to a mixture of products of gamma distributions, as $N \to \infty$. Since the expected value of $W^N([rN]+\ell)/N^2$

converges to $r(1-r)$ for all $\ell \in \mathbb{Z}$, it remains to identify its limiting distribution as the gamma with parameters 2 and $2/(r(1-r))$. This is done using the fact that the normalized covariances vanish as $N \to \infty$, as a consequence of our previous computation of the limiting covariances.

Several authors proposed mathematical models for silos. The model in which a grain lies its weight on to the lower neighbors was introduced by Harr[4] and explored by others, for example, Liu *et al.*[2] introduced the model defined by (2.1). The model with zero boundary condition was studied by Peralta-Fabi *et al.*[5] Coppersmith *et al.*[1] developed mean field computations for these models and in the uniform case, they conclude that the product of gamma distributions with parameters 2 and $2/\rho$ are invariant for a silo with periodic boundary conditions and weightless grains; they also obtained analogous results for dimensions $d \geqslant 2$. The main contribution of Theorem 2.12 is to prove that as the size N of the silo with zero boundary conditions increases, the invariant measure around a macroscopic site rN converges precisely to the gamma distribution with parameters 2 and $2/(r(1-r))$. Rajesh and Majumdar[6] compute the space-space and space-time correlations for the periodic model starting with the zero configuration. Claudin *et al.*[7] compute the spatiotemporal correlations in a continuum approximation when the mass of each grain is rescaled so that it goes to zero but Rajesh and Majumdar[6] criticize the approach. Socolar[8] shows numerical data for horizontal correlations for the model with periodic boundary conditions. Mueth *et al.*[9] show experimental data for force correlation in a silo filled with glass beads. Essentially they conclude, in agreement with the q-model, that the correlations vanish. Lewandowska *et al.*[10] computed the variances of stresses in the q model with weightless grains. Krug and Garcia[11] and Rajesh and Majumdar[12] study related conserved-mass models with periodic boundary conditions and show that in some cases the process has a stationary measure with non zero correlations.

2. DEFINITIONS AND RESULTS

To define the model let $(U_t) := \{U_t(i)\colon i \in \mathbb{Z}, t \geqslant 0\}$ be a family of independent uniform random variables in $[0, 1]$ and $(V_t) := \{V_t(i)\colon i \in \mathbb{Z}, t \geqslant 0\}$ be a family of iid positive random variables with mean 1 and variance $\alpha = \mathbb{V}V_t(i) < \infty$. Furthermore assume (V_t) and (U_t) to be independent families.

Fix $N \geqslant 1$, consider the finite box

$$\Lambda^N := \{1,..., N\}$$

and denote $W_t^N(i)$ the weight carried by a grain located at the ith position at level t. Fix an initial configuration $W_0^N \in [0, \infty)^{\Lambda^N}$ and define inductively

$$\begin{aligned} W_t^N(i) &= V_t(i) + W_{t-1}^N(i+1)\, U_{t-1}(i+1) \\ &\qquad + W_{t-1}^N(i-1)(1-U_{t-1}(i-1)), \qquad i \in \Lambda^N \\ W_t^N(0) &= W_t^N(N+1) \equiv 0. \end{aligned} \tag{2.1}$$

Let $W_t^N = (W_t^N(i) \colon i \in \Lambda^N)$; then $(W_t^N \colon t \geqslant 1)$ is a discrete time Markov chain on $[0, \infty)^{\Lambda^N}$. Each grain j of layer t gives a fraction chosen uniformly in $[0, 1]$ of its own weight plus the total weight it supports from the previous layers to grain $j-1$ of the successive layer $t+1$ (which we can think is below t) and the remaining to grain $j+1$. The weight distributed to grains outside Λ^N is thought of as being absorbed by the walls of the silo at sites 0 and $N+1$.

For a measure ν on $[0, \infty)^{\Lambda^N}$ let $\nu S^N(t)$ be the measure defined by

$$\nu S^N(t)\, f = \mathbb{E}_\nu\, f(W_t^N) := \int \nu(dW)\, \mathbb{E}(f(W_t^N) \mid W_0^N = W).$$

where $\mathbb{E}$ and $\mathbb{P}$ are the expectation and probability defined with respect to the probability space induced by $(U_t \colon t \geqslant 0)$ and $(V_t \colon t \geqslant 0)$.

We say that a measure μ^N is *invariant* for the process W_t^N if $\mu^N S^N(t) = \mu^N$.

If W_t^N has an invariant measure μ^N its mean heights $w^N(i) := \int \mu^N(dW)\, W(i)$ have to satisfy the following system of equations (taking expectations in (2.1) and assuming that the distribution of W_t^N does not depend on t):

$$\begin{aligned} w^N(i) &= 1 + \tfrac{1}{2} w^N(i-1) + \tfrac{1}{2} w^N(i+1), \qquad \text{for} \quad i \in \Lambda^N; \\ w^N(i) &= 0, \qquad \text{for} \quad i \in \mathbb{Z} \setminus \Lambda^N. \end{aligned} \tag{2.2}$$

Notice that the only finite solution $w^N(\cdot)$ of (2.2) has quadratic profile:

$$w^N(i) = i(N+1-i), \qquad \text{for} \quad i = 0,..., N+1. \tag{2.3}$$

(It is the expected time to exit Λ^N for a symmetric nearest neighbors random walk starting at i.)

Our first result says that for each N the silo with absorbing boundary conditions admits a unique invariant measure and that the process starting with any initial condition converges to the invariant measure.

Theorem 2.4. There exists a unique invariant measure μ^N for the process W_t^N defined by (2.1). The mean weights under μ^N are finite and satisfy (2.3). Furthermore, for any initial configuration W_0^N, the law of W_t^N converges to μ^N.

Denote the covariances of weights i and j under μ^N by

$$\sigma^N(i, j) := \int \mu^N(dW)\, W(i)\, W(j) - w^N(i)\, w^N(j).$$

We show the following bounds for the covariances.

Theorem 2.5. The covariances of μ^N are of order N^3: there exists a positive constant C such that for all N

$$|\sigma^N(i, j)| \leqslant CN^3, \qquad i \neq j, \quad i, j \in \Lambda_N \tag{2.6}$$

and

$$\left| \sigma^N(i, i) - \frac{(w^N(i))^2}{2} \right| \leqslant CN^3, \qquad i \in \Lambda_N. \tag{2.7}$$

Numerical analysis performed in Section 2 suggests

$$\lim_{N \to \infty} \frac{\sigma^N([xN], [yN])}{N^3} = -\frac{r(x, y)}{3};$$

$$\lim_{N \to \infty} \frac{\sigma^N([xN], [xN]) - \dfrac{(w^N([xN]))^2}{2}}{N^3} = -\frac{r(x, x)}{2}, \tag{2.8}$$

for $x, y \in [0, 1]$, $x \neq y$, where $r(x, y)$ is the unique weak solution of the PDE

$$-\Delta u - a(x, y)\left(\Delta u + 2\frac{\partial^2 u}{\partial x \partial y} \right) = f(x, y) \qquad \text{in} \quad \Omega = (0, 1) \times (0, 1)$$

$$u = 0 \qquad \text{in} \quad \partial\Omega, \tag{2.9}$$

where $a(x, x) = 1/2$ and $a(x, y) = 0$ for $x \neq y$ and Δ stands for the Laplacian operator. The right-hand-side is given by $f(x, y) = 3\sqrt{2}\,(6(x(1-x)-1) \times \delta(x-y)$, where δ is Dirac's distribution. An approximate plot of the function r obtained numerically can be seen in Fig. 1. The numerical solution also suggests that

$$\lim_{N \to \infty} \frac{\sigma^N(i, j)}{N^2} = c(i, j) > 0. \tag{2.10}$$

The meaning of (2.10) for $i = j = 1$ is that the variance of the weight discharged to the left wall at each level is of order N^2, the square of the mean. Hence the fluctuations are of order N, the order of the mean. By symmetry the same happens to the right wall. On the other hand, (2.10) for $i \neq j$ means that the stationary correlations divided by the square of the mean stay strictly positive around the origin; this discards the hypothesis that $W^N(\cdot)/N^2$ converges to a product measure. In contrast, the correlations divided by the square of the mean vanish in the interior of the silo as we show in the next theorem.

For $\rho > 0$ define

$$\nu_\rho := \text{product measure on } [0, \infty)^{\mathbb{Z}} \text{ with marginals Gamma } (2, 2/\rho) \tag{2.11}$$

(the sum of two independent exponentials with mean $\rho/2$ each). Our main result says that if W is distributed according to the invariant measure μ^N, then as $N \to \infty$, the law of W/N^2 around the site $[rN]$ converges to $\nu_{r(1-r)}$. Let τ_i be the translation operator by i on $\mathbb{Z}$ and Θ_k be the operator "divide by k": for $W \in [0, \infty)^{\mathbb{Z}}$, $(\tau_i W)(j) = W(j-i)$ and $(\Theta_k W)(j) = W(j)/k$.

Theorem 2.12. Let μ^N be the unique invariant measure for the process W_t^N. Then for each $r \in (0, 1)$

$$\lim_{N \to \infty} \tau_{[rN]} \Theta_{N^2} \mu^N = \nu_{r(1-r)} \tag{2.13}$$

weakly, where $[\cdot]$ is the integer part.

The proof of Theorem 2.12 lies on two results. The first one is related with the limit as $N \to \infty$ of the weights of the silo model divided by N^2 in sites around a macroscopic position rN. Its distribution converges to the law of another process in $(\mathbb{R}^+)^{\mathbb{Z}}$ whose evolution is the same described by (2.1) but with $V_t(i) \equiv 0$—no extra mass is added in this system. This process is called the *infinite silo model, ISM*. We follow an approach of Liggett[3] to show that in the nearest neighbors uniform case, the invariant and translation invariant measures for the ISM are mixtures of product of Gamma distributions (Proposition 4.11 later). Combining this with the fact that the stationary space-space correlations go to zero, proved in Theorem 2.5, we can single out the product measure as the limit in (2.13).

Theorems 2.4 and 2.5 are proven in Section 3. A numerical study of the corrections for the correlations in Theorem 2.5 is given at the end of Section 3. Theorem 2.12 is proven in Section 4. In Section 3 we study the infinite silo model.

3. THE FINITE STATIONARY SILO

In this section we study the silo for fixed width N. We prove the existence of an invariant measure and study its correlations.

Notice that (2.1) describes the evolution of two independent systems: $\{W_t^N(i): i+t \text{ even}\}$ and $\{W_t^N(i): i+t \text{ odd}\}$. It is convenient to keep simultaneously track of both systems.

Proof of Theorem 2.4. If we consider any initial distribution ν_0 for W_0^N, not necessarily invariant, satisfying $\mathbb{E}_{\nu_0}[W_0^N(i)] = i(N+1-i)$, $x \in \Lambda_0^N$, then for all $t \geqslant 0$

$$\mathbb{E}_{\nu_0}[W_t^N(i)] = i(N+1-i), \qquad i \in \Lambda^N. \tag{3.1}$$

Since by Markov inequality

$$\mathbb{P}_{\nu_0}[W_t^N(i) > M] \leqslant \frac{\mathbb{E}_{\nu_0}[W_t^N(i)]}{M}, \tag{3.2}$$

the sequence $(W_t: t \geqslant 0)$ is tight [this means that for all $\varepsilon > 0$ there exists a $K > 0$ such that $\mathbb{P}(W_t(i) > K) < 1 - \varepsilon$ for all i, t]. Tightness implies that the Cesaro means $\nu_T := (1/T) \sum_{t=1}^{T} S^N(t)\, \nu_0$ converge through subsequences to an invariant measure (see Liggett,[3] for instance) called μ^N.

If (t_k) is a subsequence such that $\nu_{t_k} \to \mu^N$, as $k \to \infty$, then there exist $\overline{W}_k$ with law ν_{t_k} and W with law μ^N, all defined in the same probability space, such that for $i = 1,..., N$, $\overline{W}_k(i) \to W(i)$ almost surely (Theorem 8.1, page 25 in ref. 13; this is usually called "almost sure Skorohod representation for distribution convergence"). Since the means are independent of t, Fatou's Lemma ($\mathbb{E}(\liminf_k \overline{W}_k(i)) \leqslant \liminf_k \mathbb{E}\overline{W}_k(i)$) implies that $\mathbb{E}W(i) \leqslant i(N+1-i)$, that is, the means under μ^N are bounded by the quadratic profile (2.3). But if the means are bounded and satisfy Eqs. (2.2), then μ^N must have the quadratic profile (2.3).

In order to prove uniqueness we use coupling. Construct two processes X_t and Y_t satisfying (2.1) using the same uniform random variables $(U_t(i)\colon i, t)$ and $(V_t(i)\colon i, t)$. Assume the initial configurations satisfy

$$X_0(i) \geqslant Y_0(i) \qquad \text{for} \quad i \in \Lambda^N. \tag{3.3}$$

Then it is easy to see that at any time $t \geqslant 0$

$$X_t(i) \geqslant Y_t(i) \qquad \text{for} \quad i \in \Lambda^N \tag{3.4}$$

(in this case we say that the process is *attractive*). Denote

$$D(t) := \frac{1}{N} \sum_{i=1}^{N} (X_t(i) - Y_t(i)).$$

We have

$$\begin{aligned} D(t) = D(t-1) - \frac{1-U_{t-1}(1)}{N} [X_{t-1}(1) - Y_{t-1}(1)] \\ - \frac{U_{t-1}(N)}{N} [X_{t-1}(N) - Y_{t-1}(N)]. \end{aligned} \tag{3.5}$$

The numbers $D(1), D(2),...$ form a bounded below by 0 non-increasing sequence of nonnegative random variables. Hence $D(t)$ converges almost surely. This implies that its increments must go almost surely to zero. This and (3.5) imply that

$$[X_{t-1}(1) - Y_{t-1}(1)] \to 0 \qquad \text{and} \qquad [X_{t-1}(N) - Y_{t-1}(N)] \to 0,$$

almost surely, as $t \to \infty$. Looking at the equations

$$X_{t-1}(1) - Y_{t-1}(1) = (1 - U_{t-2}(2))[X_{t-2}(2) - Y_{t-2}(2)]$$

and

$$X_{t-1}(N) - Y_{t-1}(N) = U_{t-2}(N-1)[X_{t-2}(N-1) - Y_{t-2}(N-1)],$$

it is immediate to see that

$$X_{t-2}(2) - Y_{t-2}(2) \to 0 \qquad \text{and} \qquad X_{t-2}(N-1) - Y_{t-2}(N-1) \to 0$$

almost surely, as $t \to \infty$. Using the same argument we have that for all $i \in \Lambda^N$

$$X_t(i) - Y_t(i) \to 0, \qquad \text{as} \quad t \to \infty,$$

almost surely as $t \to \infty$.

Let $(Z_0(i): i \in \Lambda^N)$ and $(W_0(i): i \in \Lambda^N)$ arbitrary random vectors (not necessarily ordered). Define

$$X_0(i) = Z_0(i) \vee W_0(i) \qquad \text{and} \qquad Y_0(i) = Z_0(i) \wedge W_0(i).$$

We have

$$Y_0(i) \leqslant W_0(i) \leqslant X_0(i).$$

Couple the four processes using Eq. (2.1). This means that the four evolutions are performed using the same random variables $V_t(i)$ and $U_t(i)$. The previous argument implies

$$W_t(i) - Y_t(i) \to 0 \qquad \text{and} \qquad X_t(i) - W_t(i) \to 0,$$

almost surely as $t \to \infty$. Since

$$W_t(i) - Y_t(i) \leqslant |W_t(i) - Z_t(i)| \leqslant X_t(i) - W_t(i),$$

we have

$$W_t(i) - Z_t(i) \to 0, \tag{3.6}$$

almost surely as $t \to \infty$.

Hence, if we pick W_0 from the distribution μ^N and Z_0 from an arbitrary invariant distribution ν, we have $\mu^N = \nu$. This proves that μ^N is the unique invariant measure. The limit (3.6) also shows that the process W_t^N starting from an arbitrary configuration converges to μ^N. ▮

Proof of Theorem 2.5. At the end of the proof we show that $|\sigma^N(i, j)| < \infty$ for all i, j and N. If W_0^N is distributed according to the invariant measure, so is W_1^N, and one can use (2.1) to show that $\sigma^N(i, j)$ satisfies the system of equations:

$$
\begin{aligned}
\sigma^N(i,i) &= \alpha + \tfrac{1}{3}\sigma^N(i+1,i+1) + \tfrac{1}{3}\sigma^N(i-1,i-1) + \tfrac{1}{4}\sigma^N(i-1,i+1)\\
&\quad + \tfrac{1}{4}\sigma^N(i+1,i-1) + \tfrac{1}{12}(w^N(i+1))^2 + \tfrac{1}{12}(w^N(i-1))^2, \qquad i \in \Lambda^N\\
\sigma^N(i,i+2) &= \tfrac{1}{4}\sigma^N(i+1,i+3) + \tfrac{1}{4}\sigma^N(i-1,i+3) + \tfrac{1}{4}\sigma^N(i-1,i+1)\\
&\quad + \tfrac{1}{6}\sigma^N(i+1,i+1) - \tfrac{1}{12}(w^N(i+1))^2, \qquad i \in \{1,\dots,N-2\}\\
\sigma^N(i,i-2) &= \tfrac{1}{4}\sigma^N(i+1,i-1) + \tfrac{1}{4}\sigma^N(i-1,i-3) + \tfrac{1}{4}\sigma^N(i+1,i-3)\\
&\quad + \tfrac{1}{6}\sigma^N(i-1,i-1) - \tfrac{1}{12}(w^N(i-1))^2, \qquad i \in \{3,\dots,N\}\\
\sigma^N(i,j) &= \tfrac{1}{4}\sigma^N(i+1,j+1) + \tfrac{1}{4}\sigma^N(i-1,j-1) + \tfrac{1}{4}\sigma^N(i-1,j+1)\\
&\quad + \tfrac{1}{4}\sigma^N(i+1,j-1), \qquad i,j \in \Lambda^N, \quad |i-j| \geqslant 2\\
\sigma^N(i,j) &= 0, \qquad i \in \{0, N+1\} \quad \text{or} \quad j \in \{0, N+1\}.
\end{aligned} \tag{3.7}
$$

Define, for $i \in \{0, 1,\dots, N+1\}$:

$$
\begin{aligned}
R(i,i) &= -2\sigma^N(i,i) + (w^N(i))^2,\\
R(i,j) &= -3\sigma^N(i,j) \qquad \text{for} \quad i \neq j.
\end{aligned} \tag{3.8}
$$

The system (3.7) translates into the following system for the matrix R:

$$
\begin{aligned}
R(i,j) &= K(i)\, \mathbf{1}\{i=j\} + \sum_{(i',j')} p((i,j),(i',j'))\, R(i',j') \qquad i,j \in \Lambda^N,\\
R(i,j) &= 0 \qquad i \in \{0, N+1\} \quad \text{or} \quad j \in \{0, N+1\},
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
K(i) &= (w^N(i))^2 - \frac{(w^N(i+1))^2}{2} - \frac{(w^N(i-1))^2}{2} - 2\alpha\\
&= 6i(N+1-i) - (N+1)^2 - (1+2\alpha),
\end{aligned} \tag{3.10}
$$

for $i \in \Lambda^N$ and

$$
p((i,j),(i',j')) := \begin{cases} 1/4, & \text{if} \quad i \neq j \text{ and } |i-i'| \cdot |j-j'| = 1\\ 1/3, & \text{if} \quad i = j,\, i' = j' \text{ and } |i-i'| = 1\\ 1/6, & \text{if} \quad i = j,\, i' \neq j' \text{ and } |i-i'| \cdot |j-j'| = 1\\ 0 & \text{otherwise.} \end{cases} \tag{3.11}
$$

for $i, j \in \{1,..., N\}$ and

$$p((i,j),(i',j')) := \begin{cases} 1 & \text{if} \quad (i',j') = (i,j) \\ 0 & \text{otherwise} \end{cases}$$

if $i \in \{0, N+1\}$ or $j \in \{0, N+1\}$ (this means absorbing borders). Let $X_n^{(i,j)}$ be a Markov chain with transition probabilities $p(\cdot,\cdot)$ and initial state (i, j). Defining

$$T^{(i,j)} := \sum_{n \geqslant 0} \sum_{\ell=1}^{N} \mathbf{1}\{X_n^{(i,j)} = (\ell,\ell)\} \, K(\ell) \tag{3.12}$$

we have that $\mathbb{E}T^{(i,j)}$ is the unique finite solution of (3.9). Assuming that $|\sigma^N(i,j)|$ is finite, then $|R(i,j)|$ is also finite and $R(i,j) = \mathbb{E}T^{(i,j)}$.

To get an upperbound for $R(i,j)$ we consider a random walk $\tilde{X}_n^{(i,j)}$ on the diamond

$$\mathbf{\Delta}^N := \{(i,j) \in \mathbb{Z}^2 : i+j \text{ even}, 2 \leqslant i+j \leqslant 2N, |i-j| \leqslant N+1\}$$

with transition probabilities

$$\begin{aligned} &\tilde{p}((i,j),(i',j')) = p((i,j),(i',j')) \text{ given by (3.11)} \\ &\qquad \text{if } (i,j) \in \{(i,j) \in \Lambda^N \times \Lambda^N : 4 \leqslant i+j \leqslant 2N-4, |i-j| \leqslant N-1\} \end{aligned} \tag{3.13}$$

(the interior of the diamond); periodic boundary conditions along two of the sides of the diamond:

$$\begin{aligned} &\tilde{p}((i,j),(N+1-j, N+1-i)) = \tilde{p}((N+1-j, N+1-i),(i,j)) = 1/4 \\ &\qquad \text{if} \quad i+j=2, \quad i \neq j \\ &\tilde{p}((i,j),(i',j')) = \tilde{p}((N+1-j, N+1-i),(N+1-j', N+1-i')) = 1/4 \\ &\qquad \text{if} \quad i+j=2, \quad i \neq j, \quad |i-i'| \cdot |i-j'| = 1, \quad (i',j') \in \mathbf{\Delta}^N \end{aligned}$$

$$\tilde{p}((1,1),(i,j)) = \begin{cases} 1/3 & \text{if } (i,j) \in \{(2,2),(N,N)\} \\ 1/6 & \text{if } (i,j) \in \{(2,0),(0,2)\} \end{cases} \tag{3.14}$$

$$\tilde{p}((N,N),(i,j)) = \begin{cases} 1/3 & \text{if } (i,j) \in \{(1,1),(N-1,N-1)\} \\ 1/6 & \text{if } (i,j) \in \{(N-1,N+1),(N+1,N-1)\} \end{cases} \tag{3.15}$$

and absorbing probabilities along the other two sides:

$$\tilde{p}((i,j),(i,j))=1, \qquad (i,j)\in\Delta^N, \quad |i-j|=N+1. \tag{3.16}$$

The point of this random walk is that it is "one dimensional" in the sense that the one-dimensional process defined by $Z_n^{i-j} := (\tilde{X}_n^{(i,j)})_1 - (\tilde{X}_n^{(i,j)})_2$ (the difference of the coordinates of $\tilde{X}_n^{(i,j)}$) is Markovian and has transition probabilities

$$p_0(i,j)=\begin{cases} 1/4 & \text{if} \quad i\notin\{-N-1,0,N+1\} \quad \text{and} \quad |i-j|=2 \\ 1/2 & \text{if} \quad j=i \quad \text{and} \quad i\notin\{-N-1,0,N+1\} \\ 2/3 & \text{if} \quad i=j=0 \\ 1/6 & \text{if} \quad i=0 \quad \text{and} \quad |i-j|=2 \\ 1 & \text{if} \quad i=j \quad \text{and} \quad i\in\{-N-1,N+1\} \end{cases} \tag{3.17}$$

for even $i\in\{-N-1,\ldots,N+1\}$.

Define $\bar{K}=\max_\ell K(\ell)$, $\underline{K}=\min_\ell K(\ell)$. We have $\underline{K}<0<\bar{K}$ and both $\underline{K}$, $\bar{K}$ are of order of N^2. Define

$$\bar{T}^{(i,j)} := \sum_{n\geqslant 0}\sum_{\ell=1}^{N} \mathbf{1}\{\tilde{X}_n^{(i,j)}=(\ell,\ell)\}\,\bar{K}$$

$$\underline{T}^{(i,j)} := \sum_{n\geqslant 0}\sum_{\ell=1}^{N} \mathbf{1}\{\tilde{X}_n^{(i,j)}=(\ell,\ell)\}\,\underline{K}.$$

A simple coupling argument shows that for all $i,j\in\Lambda^N$,

$$\underline{T}^{(i,i)}\leqslant \underline{T}^{(i,j)}\leqslant T^{(i,j)}\leqslant \bar{T}^{(i,j)}\leqslant \bar{T}^{(i,i)}. \tag{3.18}$$

On the other hand $\underline{T}^{(i,i)}$ and $\bar{T}^{(i,i)}$ do not depend on i and can be expressed in function of a geometric random sum of geometric random variables:

$$\underline{T}^{(i,i)}=\underline{K}\sum_{k=1}^{S} M_k; \qquad \bar{T}^{(i,i)}=\bar{K}\sum_{k=1}^{S} M_k, \tag{3.19}$$

where $\mathbb{P}(S\geqslant n)=(\frac{N-1}{N+1})^{n-1}$, $n\geqslant 1$ and $\mathbb{P}(M_k\geqslant m)=(2/3)^{m-1}$, $m\geqslant 1$. Furthermore (M_k) are iid and independent of S. The random variable S counts the number of times the chain Z_n starting at 2 or -2 comes back to zero before arriving to $N+1$ or $-N-1$ while M_k is the time the chain Z_n spends in its kth visit to the origin. Taking expectations in (3.19) and using Wald identity we get

$$\mathbb{E}\bar{T}^{(i,i)}=\tfrac{3}{2}(N+1)\,\bar{K}; \qquad \mathbb{E}\underline{T}^{(i,i)}=\tfrac{3}{2}(N+1)\,\underline{K}. \tag{3.20}$$

Taking expectations in (3.18) and using the fact that both $\underline{K}$ and $\bar{K}$ are of order N^2, we have proved that if $R(i, j)$ is finite, then

$$-CN^3 \leqslant \underline{R}(i, j) \leqslant R(i, j) \leqslant \bar{R}(i, j) \leqslant CN^3. \tag{3.21}$$

This, (3.8) and (3.1) show (2.4) and (2.5).

Finiteness of $\sigma^N(i, j)$. Let W_0^N be a non negative random configuration chosen according to a product measure with marginals satisfying

$$\begin{aligned} \mathbb{E}W_0^N(i) &= w^N(i) \\ \mathbb{V}W_0^N(i) &= (w^N(i))^2/2 \end{aligned} \tag{3.22}$$

(the covariances are null). The covariances at time t are given by

$$\sigma_t^N(i, j) = \begin{cases} \frac{1}{2}(w^N(i))^2 - \frac{1}{2}R_t^0(i, i) & \text{if} \quad i = j \\ -\frac{1}{3}R_t^0(i, j) & \text{if} \quad i \neq j, \end{cases} \tag{3.23}$$

where $R_t^0(i, j)$ evolves according to

$$R_t^0(i, j) = \mathbf{1}\{i = j\}\ K(i) + \sum_{(i', j')} p((i, j), (i', j'))\ R_{t-1}^0(i', j'), \tag{3.24}$$

with $R_0^0(i, j) = 0$ for $i, j \in \{0,..., N+1\}$. Let $\bar{R}_t$ and $\underline{R}_t^0$ be the systems defined in the diamond $\mathbf{\Delta}^N$ with initial condition $\bar{R}_0^0(i, j) = \underline{R}_0^0(i, j) = 0$, $i, j \in \mathbf{\Delta}^N$ and evolving with the equations

$$\bar{R}_t^0(i, j) = \mathbf{1}\{i = j\}\ \bar{K} + \sum_{(i', j')} \tilde{p}((i, j), (i', j'))\ \bar{R}_{t-1}^0(i', j'). \tag{3.25}$$

$$\underline{R}_t^0(i, j) = \mathbf{1}\{i = j\}\ \underline{K} + \sum_{(i', j')} \tilde{p}((i, j), (i', j'))\ \underline{R}_{t-1}^0(i', j'). \tag{3.26}$$

It is clear that

$$\underline{R}_t^0(i, j) \leqslant R_t^0(i, j) \leqslant \bar{R}_t^0(i, j).$$

On the other hand

$$\bar{R}_t^0(i, j) \leqslant \bar{R}_t(i, j) \leqslant CN^3, \tag{3.27}$$

where $\bar{R}_t \equiv \bar{R}$ is the stationary solution of (3.25). That is, $\bar{R}_t^0(i, j)$ is bounded by the stationary solution of (3.25) which in turn is bounded as in

(3.21). The same argument shows that $|\underline{R}_t^0(i,j)|$ is uniformly bounded in t. Hence $|R_t^0(i,j)|$ is uniformly bounded in t and by (3.23) so is $|\sigma_t^N(i,j)|$. To show the finiteness of $\sigma^N(i,j)$; we need to take the limit as $t\to\infty$. Since the distribution ν_t^N of $W_t^N(i)$, converges weakly to μ^N, as $t\to\infty$, we can again use Theorem 8.1, page 25 in ref. 13 to define $W_t^N(i)$ and $W^N(i)$ in the same space with marginal laws ν_t^N and μ^N, respectively, such that $W_t^N(i)$ converges almost surely to $W^N(i)$, as $t\to\infty$. Now, $\sigma^N(i,i)=\mathbb{V}W^N(i)=\mathbb{E}(W^N(i))^2-(\mathbb{E}W^N(i))^2$. Since $\mathbb{E}W^N(i)=\mathbb{E}W_t^N(i)$ for all t, Fatou's Lemma (which in this case says $\mathbb{E}(\liminf_t(W_t^N(i))^2)\leqslant\liminf_t\mathbb{E}(W^N(i))^2$) implies

$$\sigma^N(i,i)\leqslant\liminf_{t\to\infty}\sigma_t^N(i,i)\leqslant\frac{(w^N(i))^2}{2}+CN^3 \tag{3.28}$$

by (3.23) and (3.21). This settles the finiteness of the diagonal covariances. By (3.7), also the other covariances must be finite. ▮

Numerical Analysis of the Equation for the Correction R. We carried out a numerical investigation about the solution of Eq. (3.9). This equation can be seen as a discretized version of the partial differential equation (2.9) in the following way. We first divide system (3.9) by $(N+1)^3$, obtaining a system for the new variable $r_{i,j}:=R_{i,j}/(N+1)^3$, which is to be seen as an approximation to the value of a function $r(x,y)$ defined on $\Omega=[0,1]\times[0,1]$ at the point $(x_i,y_j)=(ih,jh)$, where $h=1/(N+1)^3$. The equations for $i\neq j$ are of the form

$$4r_{i,j}-r_{i-1,j-1}-r_{i-1,j+1}-r_{i+1,j-1}-r_{i+1,j+1}=0$$

which is a discrete form of $\Delta r=0$, where Δ denotes the (rotation invariant) Laplace operator. Now, for $i=j$ we multiply (3.9) by $3h=3/(h^2(N+1)^3)$ and obtain the equations:

$$\begin{aligned}&\frac{1}{2h^2}(4r_{i,i}-r_{i-1,i-1}-r_{i-1,i+1}-r_{i+1,i-1}-r_{i+1,i+1})\\&\quad+\frac{1}{2h^2}(2r_{i,i}-r_{i-1,i-1}-r_{i+1,i+1})\\&=3\sqrt{2}\,(6x_i(1-x_i)-1-(1+2\alpha)\,h^2)\,\delta_{\sqrt{2}h}(i-i),\end{aligned} \tag{3.29}$$

where $x_i=ih$ and $\delta_{\sqrt{2}h}$ is a piecewise linear approximation to Dirac's distribution, assuming the value $1/(\sqrt{2}\,h)$ at the origin, and being zero at the other points of an equally spaced mesh (with mesh-space $H=\sqrt{2}\,h$). Now, we observe that the first term in Eq. (3.29) is a discretization of minus the

Laplacian of r (in rotated form) and that the second expression corresponds to minus the second derivative of r along the diagonal $x = y$ (which can be written as $-1/2\Delta r - \partial^2 r/\partial x \partial y$). The right-hand-side of (3.29) approximates the function $3\sqrt{2}(6x(1-x)-1)\,\delta(x-y)$ over the diagonal. All together we obtain Eq. (2.9), which shall be interpreted in the weak sense.

To compute the solution of Eq. (3.9) we have developed a multigrid method composed by red-black Gauss–Seidel relaxations, full-weighting of residuals for fine-to-coarse mesh transfers and of bilinear interpolation for coarse-to-fine corrections (see, e.g., Hackbusch[14] for definitions). The scheme is very fast, it takes only a few seconds to solve Eq. (3.9) for $N = 511$ on a Sun-workstation and around 15 minutes for the solution for $N = 4095$ on a Dec-alpha (the equation in this case has more than 16 million unknowns). The equations were solved to the level of machine precision, using double precision (16-byte) words.

The results shown in Figs. 1–3 were obtained with $\alpha = 0$, but the same asymptotics are expected for other values of α. In Fig. 1 we present the function $r = R/(N+1)^3$, with a resolution of $N = 127$ (the function doesn't change much visually for larger values of N). In Fig. 2 we plot $r = R/(N+1)^3$ over the diagonal $\{i = j\}$, for several values of N up to 511 —for larger values the graphics are indistinguishable. We can observe the convergence of r as N grows (to the weak solution of the partial differential

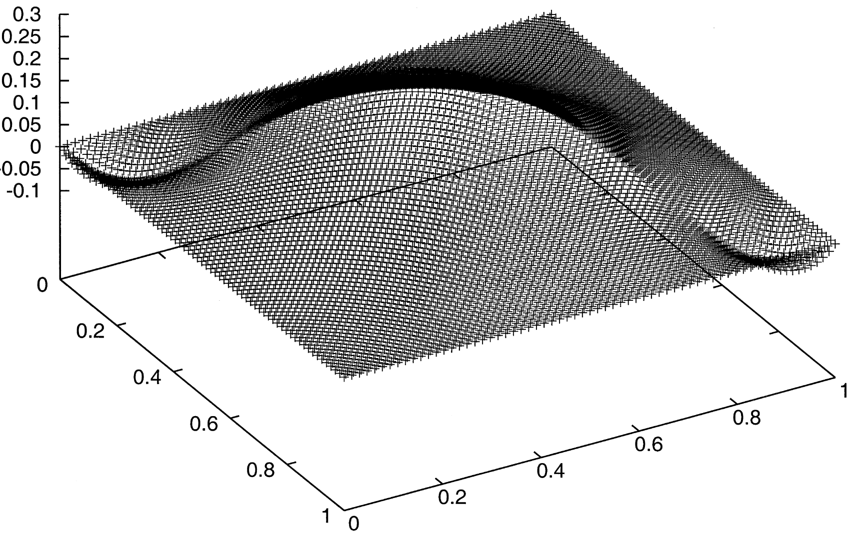

Fig. 1. Graphic of the computed solution $r = R/(N+1)^3$ for $N = 127$.

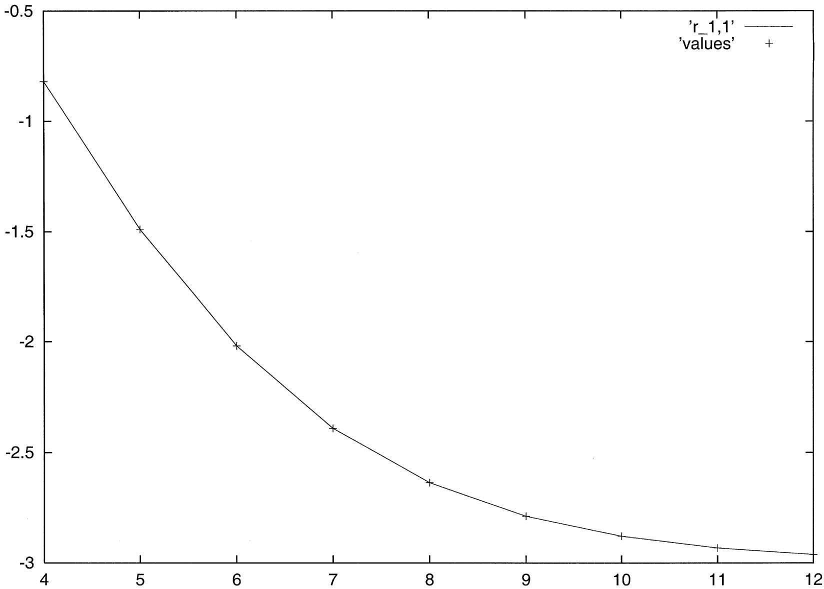

Fig. 2. Graphic of the computed solution $r = R/(N+1)^3$ over the diagonal $(i = j)$ for several values of N.

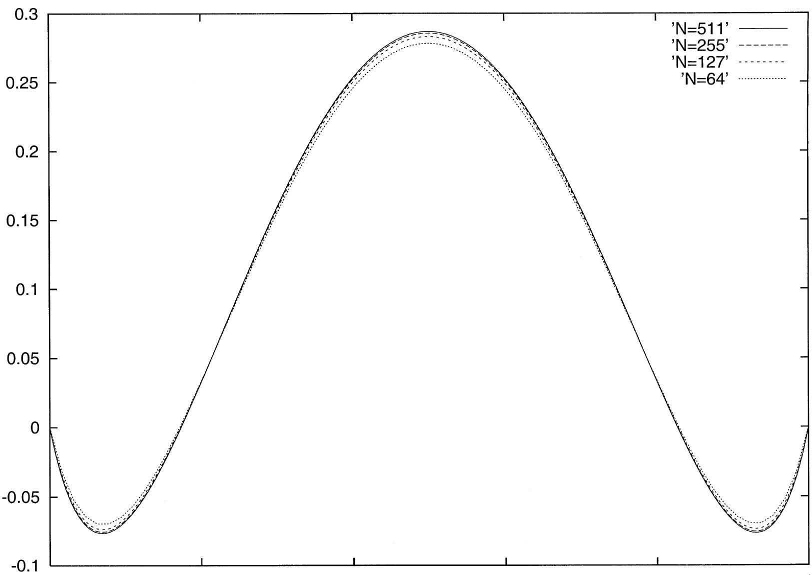

Fig. 3. Graphic of the values of $R(1, 1)/(N+1)^2$ for several values of $\log_2(N+1)$.

equation (2.9)). The upper and lower constant bounds for $R/(N+1)^3$ we proved to exist in Theorem 2.5, are attained over the diagonal $\{i=j\}$ and their approximate values can be read from Fig. 2.

Another aspect we tried to access from the numerical solutions is the behaviour of $R(1,1)/(N+1)^2$ (which corresponds to the derivative of r over the diagonal—the function of Fig. 2—at $x=0$). We computed these values for $N=15$ up to $N=4095$ (with $N+1$ as powers of two). The results are presented graphically in Fig. 3, where N is displayed in log scale. The results suggest that as $N\to\infty$ $R(1,1)/(N+1)^2$ converges to a constant (around -3). This together with the definition of σ^N given in (3.8) supports (2.10). The constant $c(1,1)$ seems to be around 4.

4. THE INFINITE SILO MODEL

To prove Theorem 2.12 we will use a spatially infinite limit of the silo model. This model represents the transmission of weight among weightless grains in an infinitely large silo. For each $i\in\mathbb{Z}$ let $\eta_t(i)$ denote the weight supported by site i at time $t\geqslant 0$; the dynamics is represented by the system of equations: for $i\in\mathbb{Z}$ and $t\geqslant 1$ such that it is even,

$$\eta_t(i)=\eta_{t-1}(i-1)\,U_{t-1}(i-1)+\eta_{t-1}(i+1)[1-U_{t-1}(i+1)], \tag{4.1}$$

where $(U_t(i): i\in\mathbb{Z}, t\geqslant 1)$ are i.i.d. uniform random variables in $[0,1]$. We call this process *infinite silo model* (ISM). Let ν_ρ be a product measure on $[0,\infty)^{\mathbb{Z}}$ with marginals Gamma $(2, 1/(2\rho))$; ρ is the expected value of each marginal. Coppersmith et al. proved that the ν_ρ is invariant for the process with periodic boundary conditions. We show now that ν_ρ is reversible (and hence invariant) for the infinite system.

Proposition 4.2. For each $\rho\geqslant 0$ the measure ν_ρ is reversible for the ISM, that is, for all cylinder functions f and g, $\mathbb{E}_{\nu_\rho}(f(\eta_0)\,g(\eta_t))=\mathbb{E}_{\nu_\rho}(g(\eta_0)\,f(\eta_t))$.

The proof is based on the following elementary lemma.

Lemma 4.3. Let W be a random variable with distribution Gamma $(2,\rho/2)$. Let U be a uniform random variable in $[0,1]$ independent of W. Then, UW and $(1-U)\,W$ are independent exponentially distributed random variables with mean $\rho/2$.

Proof of Proposition 4.2. It suffices to prove that if η_0 has law ν_ρ then, for cylinder functions f and g,

$$\mathbb{E}f(\eta_0)\,g(\eta_1)=\mathbb{E}g(\eta_0)\,f(\eta_1).$$

Let

$$X(i) = U_0(i)\,\eta_0(i); \qquad Y(i) = (1 - U_0(i))\,\eta_0(i).$$

Under ν_ρ, $(\eta_0(i): i \in \mathbb{Z})$ is a family of iid gammas random variables. Then, by the lemma, $(X(i): i \in \mathbb{Z})$ and $(Y(i): i \in \mathbb{Z})$ are independent families of iid exponential random variables of mean $\rho/2$ and

$$\mathbb{E} f(\eta_0)\, g(\eta_1) = \mathbb{E} f(X+Y)\, g(X'+Y'), \tag{4.4}$$

where $(X'(i), Y'(i)) := H(X, Y)(i) := (Y(i-1), X(i+1))$. Since $H(H(X, Y)) = (X, Y)$ and (X', Y') has the same law as (X, Y) (iid exponential random variables), (4.4) equals to

$$\mathbb{E} f(X'+Y')\, g(X+Y) = \mathbb{E} g(\eta_0)\, f(\eta_1). \quad \blacksquare$$

One of the tools to prove Theorem 2.12 is to show that all invariant and translation invariant measures for the ISM can be written as convex combinations of those product measures.

For any two initial configurations η and ξ, we couple the two processes η_t and ξ_t using the same uniform random variables to update the process. Notice that this coupling is attractive, that is,

$$\eta(i) \leqslant \xi(i) \ \text{ for all } i \in \mathbb{Z}, \ \text{ then } \eta_t(i) \leqslant \xi_t(i), \ \text{ for all } i \in \mathbb{Z}. \tag{4.5}$$

Through out this section $\mathscr{I}$ and $\mathscr{T}$ ($\bar{\mathscr{I}}$ and $\bar{\mathscr{T}}$) will denote, respectively, the set of invariant and translation invariant measures for the process (coupled process) defined by (4.1).

Proposition 4.6. If $\nu \in \bar{\mathscr{I}} \cap \bar{\mathscr{T}}$, then

$$\nu\{(\eta, \xi);\, \eta \geqslant \xi \text{ or } \eta \leqslant \xi\} = 1. \tag{4.7}$$

The proof of this proposition is based on the following lemma.

Lemma 4.8. Let X, Y be identically distributed real random variables defined in the same probability space. Let U and V be identically distributed and independent random variables in $[0, 1]$ with $\mathbb{E}U = 1/2$; U and V are assumed to be independent of X and Y. If $Z = UX + VY$ is distributed as X (and Y), then $\mathbb{P}(XY \geqslant 0) = 1$.

Proof. From the definition of Z,

$$\begin{aligned}
\mathbb{E}(Z\mathbf{1}\{Z\geqslant 0\}) &= \mathbb{E}(UX\mathbf{1}\{Z\geqslant 0\})+\mathbb{E}(VY\mathbf{1}\{Z\geqslant 0\})\\
&= 2\mathbb{E}(UX\mathbf{1}\{Z\geqslant 0\})\\
&= 2\mathbb{E}(UX\mathbf{1}\{Z\geqslant 0,\, X\geqslant 0\})+2\mathbb{E}(UX\mathbf{1}\{Z\geqslant 0,\, X<0\})\\
&= 2\mathbb{E}(UX\mathbf{1}\{X\geqslant 0\})-2\mathbb{E}(UX\mathbf{1}\{X\geqslant 0,\, Z<0\}))\})\\
&\quad +2\mathbb{E}(UX\mathbf{1}\{Z\geqslant 0,\, X<0\}).
\end{aligned}$$

The second identity follows from the fact that $UX\mathbf{1}\{Z\geqslant 0\}$ and $VY\mathbf{1}\{Z\geqslant 0\}$ have the same law. The third and fourth identities come from set operations. Since X and Z are identically distributed and U is independent of X, $\mathbb{E}(Z\mathbf{1}\{Z\geqslant 0\})=2\mathbb{E}(UX\mathbf{1}\{X\geqslant 0\})$. Canceling those terms we get

$$0=-\mathbb{E}(UX\mathbf{1}\{X\geqslant 0,\, Z<0\}))\})+\mathbb{E}(UX\mathbf{1}\{Z\geqslant 0,\, X<0\}).$$

This implies

$$\begin{aligned}
\mathbb{P}(X>0,\, Z<0) &= \mathbb{P}(X<0,\, Z\geqslant 0)=0\\
\mathbb{P}(Y>0,\, Z<0) &= \mathbb{P}(Y<0,\, Z\geqslant 0)=0
\end{aligned} \tag{4.9}$$

because (X, Z) and (Y, Z) have the same distribution. Hence

$$\mathbb{P}(X>0,\, Y<0)=\mathbb{P}(X>0,\, Y<0,\, Z\geqslant 0)+\mathbb{P}(X>0,\, Y<0,\, Z<0)=0.$$

The same argument shows that $\mathbb{P}(X<0,\, Y>0)=0$. ∎

Proof of Proposition 4.6. Let (η_0, ξ_0) be distributed according to $\nu \in \bar{\mathscr{I}} \cap \bar{\mathscr{T}}$. Then the variables

$$X=\xi_0(i-1)-\eta_0(i-1);\qquad Y=\xi_0(i+1)-\eta_0(i+1);\qquad Z=\xi_1(i)-\eta_1(i) \tag{4.10}$$

satisfy the hypothesis of Lemma 4.8. Hence, for all $i \in \mathbb{Z}$,

$$\begin{aligned}
&\nu(\xi(i-1)=\eta(i-1))=1 \qquad \text{or}\\
&\nu([\xi_0(i-1)-\eta_0(i-1)]\,[\xi_0(i+1)-\eta_0(i+1)]>0)=1,
\end{aligned}$$

which implies the proposition. ∎

Proposition 4.11. If $\mu \in (\mathscr{I} \cap \mathscr{T})$ then

$$\mu = \int \nu_\rho \, d\lambda(\rho), \tag{4.12}$$

where λ is a probability measure on (0, 1).

Proof. For each $\rho \geqslant 0$ we can construct $\bar{\nu}_\rho \in \bar{\mathscr{I}} \cap \bar{\mathscr{T}}$, a measure on $[0, \infty)^{\mathbb{Z}} \times [0, \infty)^{\mathbb{Z}}$ with marginals μ and ν_ρ (Liggett, ref. 3).

We claim that $\mu \in \mathscr{I} \cap \mathscr{T}$ concentrate its mass on configurations with asymptotic density. That is, for a configuration η define

$$M_n(\eta) = \frac{1}{2n+1} \sum_{i=-n}^{n} \eta(i). \tag{4.13}$$

Then, for all $a \in [0, 1]$,

$$\mu\{\eta; \liminf_{n \to \infty} M_n(\eta) < a < \limsup_{n \to \infty} M_n(\eta)\} = 0. \tag{4.14}$$

In fact, if (4.8) is false, there exists an a such that

$$\bar{\nu}_a\{(\eta, \xi); \eta \geqslant \xi \text{ or } \eta \leqslant \xi\}^c \geqslant \mu\{\eta; \liminf M_n(\eta) < a < \limsup M_n(\eta)\} > 0, \tag{4.15}$$

which contradicts Proposition 4.6. Therefore, taking $0 = \rho_0 < \rho_1 < \cdots < \rho_m = 1$, with $\rho_i = i/m$, with m positive integer, we have

$$\mu(\cdot) = \sum_{i=1}^{m} \mu(\cdot \mid C_i) \, p_i, \tag{4.16}$$

where $C_i := \{\eta; \lim_n M_n(\eta) \in (\rho_{i-1}, \rho_i]\}$ and $p_i = \mu(C_i)$, for $i = 1, \ldots, m$.

Define

$$\underline{\mu}_m(\cdot) = \sum_{i=1}^{m} \nu_{\rho_{i-1}} \, p_i; \qquad \bar{\mu}_m(\cdot) = \sum_{i=1}^{m} \nu_{\rho_i} \, p_i. \tag{4.17}$$

By definitions (4.16) and (4.17) and Proposition 4.6 we can couple μ, $\underline{\mu}$ and $\bar{\mu}$ and show

$$\underline{\mu}_m \leqslant_{st} \mu \leqslant_{st} \bar{\mu}_m.$$

for all m, where $\leqslant_{st}$ is the usual stochastic domination of measures defined by: $\mu \leqslant_{st} \nu$ if and only if for all non decreasing function f, $\mu f \leqslant \mu' f$; equivalently, $\mu \leqslant_{st} \mu'$ if and only if there exists a joint realization (W, W') with marginals μ and μ' such that $W \leqslant W'$. Defining $\lambda(\rho) = \mu(\eta\colon \lim_n M_n(\eta) \leqslant \rho)$ we have

$$\lim_{m\to\infty} \underline{\mu}_m = \lim_{m\to\infty} \bar{\mu}_m = \int \nu_\rho \, d\lambda(\rho)$$

This proves the proposition. ∎

5. PROOF OF THEOREM 2.12

In this section we use the characterization of the set of invariant and translation invariant measures for the ISM given by Proposition 4.11 and the bounds on the covariances (2.6) to prove Theorem 2.12. Let μ^N be the unique invariant measure for the silo in the finite box Λ^N. We extend the definition of the measure μ^N in $[0, \infty)^{\mathbb{Z}}$ by setting $\mu^N(\eta\colon \eta(i) = 0) = 1$ for all $i \notin \Lambda^N$.

Let $U_t(i)$ be the uniform random variables defined in the introduction and define $\mathbf{U}_t$ as the matrix with entries

$$\mathbf{U}_t(i, j) := \begin{cases} U_t(i) & \text{if} \quad j = i-1 \\ 1 - U_t(i) & \text{if} \quad j = i+1 \\ 0 & \text{if} \quad |j-i| > 1. \end{cases} \tag{5.1}$$

Define the truncations

$$\begin{aligned} [\mathbf{U}]^N(i, j) &:= \mathbf{U}(i, j) \, \mathbf{1}\{i \in \Lambda^N\} \\ [V]^N(i) &:= V(i) \, \mathbf{1}\{i \in \Lambda^N\}. \end{aligned} \tag{5.2}$$

The evolution equation (2.1) is then a product of a left vector and a matrix plus an independent vector:

$$W^N_{t+1} = W^N_t [\mathbf{U}_t]^N + [V_{t+1}]^N.$$

Notice that the definition of $[\mathbf{U}_t]^N$ guarantees $W^N_t(j) \equiv 0$ for $j \notin \Lambda^N$.

Existence of the Limit. Let W^N be a configuration distributed according to the invariant measure μ^N. The maximum of (2.3) is attained for $i \in \{N/2, (N+1)/2\}$, hence

$$\mathbb{E}\left[\frac{W^N([rN]+\ell)}{N^2}\right] \leqslant \frac{1}{4}+O\left(\frac{1}{2N}\right), \tag{5.3}$$

for all $\ell \in \mathbb{Z}$. Therefore, the sequence of probability measures $\{\tau_{[rN]}\Theta_{N^2}\mu^N, N \geqslant 1\}$ (that is, $\{$law of $\frac{W^N([rN]+\cdot)}{N^2}, N \geqslant 1\}$) is tight. Consider a convergent subsequence $\{N_k, k \geqslant 1\}$ and let μ be the weak limit of this subsequence:

$$\mu := \lim_{k \to \infty} \tau_{[rN_k]}\Theta_{N_k^2}\mu^{N_k}. \tag{5.4}$$

Invariance of the Limit. The equation $\mu^N = S^N(t)\,\mu^N$ is equivalent to

$$\mathbb{E}f(W^N) = \mathbb{E}f(W^N[\mathbf{U}]^N + [V]^N), \tag{5.5}$$

for cylinder bounded f, where $\mathbf{U}$ has the same law as $\mathbf{U}_t$, V is a vector with the same law as V_{t+1} and W^N, $\mathbf{U}$ and V are independent.

If η is distributed according to ν, then the invariance of ν for the infinite silo model is equivalent to

$$\mathbb{E}f(\eta) = \mathbb{E}f(\eta\mathbf{U}), \tag{5.6}$$

for cylinder bounded f, where η and $\mathbf{U}$ are independent.

Let $\tilde{\eta}^k = \tau_{[rN_k]}\Theta_{N_k^2}W^{N_k}$, $\tilde{V}^k = \tau_{[rN_k]}V$ and $\tilde{\mathbf{U}}^k = \tau_{[rN_k]}\mathbf{U}$, where $(\tau_\ell\mathbf{U})(i, j) = \mathbf{U}(i-\ell, j-\ell)$. Take k so large such that the support of $\tau_{[rN_k]}f$ is inside $\Lambda^{N_k}\setminus\{1, N_k\}$ (and the truncations are unnecessary). Then the law of $\tilde{\eta}^k$ must satisfy

$$\mathbb{E}f(\tilde{\eta}^k) = \mathbb{E}f(\tilde{\eta}^k\tilde{\mathbf{U}}^k + (\tilde{V}^k/N_k^2)). \tag{5.7}$$

By translation invariance, $\tilde{\mathbf{U}}^k$ and $\mathbf{U}$ have the same law, $\tilde{V}^k$ and V have the same law and by (5.4) $\tilde{\eta}^k$ converges in distribution to μ as $k \to \infty$; hence μ must satisfy (5.6), that is, $\mu \in \mathscr{I}$.

Translation Invariance of the Limit. Let $(W_0^{N-2}, W_0^N, W_0^{N+2})$ be identically null and

$$\begin{aligned}(W_{t+1}^{N-2}, W_{t+1}^N, W_{t+1}^{N+2}) &= (W_t^{N-2}[\tau_1\mathbf{U}_t]^{N-2} + [\tau_1 V_{t+1}]^{N-2},\\ &\quad W_t^N[\tau_1\mathbf{U}_t]^N + [\tau_1 V_{t+1}]^N, W_t^{N+2}[\mathbf{U}_t]^{N+2} + [V_{t+1}]^{N+2})\end{aligned} \tag{5.8}$$

for $t \geqslant 0$. Then $W_t^{N-2} \leqslant W_t^N \leqslant W_t^{N+2}$ for all t. By Theorem 2.4 (W_t^{N-2}, W_t^N, W_t^{N+2}) converges in distribution to (W^{N-2}, W^N, W^{N+2}) a vector with marginals $\tau_1\mu^{N-2}$, $\tau_1\mu^N$ and μ^{N+2} also satisfying $W^{N-2} \leqslant_{st} W^N \leqslant_{st} W^{N+2}$, where $X \leqslant_{st} Y$ means that the law of X is stochastically dominated by the law of Y (see the definition at the end of Section 4). This implies

$$\tau_1\mu^{N-2} \leqslant_{st} \tau_1\mu^N \leqslant_{st} \mu^{N+2}. \tag{5.9}$$

Analogously, the coupling

$$\begin{aligned}(W_{t+1}^{N-2}, W_{t+1}^N, W_{t+1}^{N+2}) \\ &= (W_t^{N-2}[\tau_1\mathbf{U}_t]^{N-2} + [\tau_1 V_{t+1}]^{N-2}, \\ &\quad W_t^N[\mathbf{U}_t]^N + [V_{t+1}]^N, W_t^{N+2}[\mathbf{U}_t]^{N+2} + [V_{t+1}]^{N+2})\end{aligned} \tag{5.10}$$

shows that

$$\tau_1\mu^{N-2} \leqslant_{st} \mu^N \leqslant_{st} \mu^{N+2}. \tag{5.11}$$

From the coupling (5.10) and by Chebyshev,

$$\mathbb{P}\left(\frac{W^{N+2}(i) - W^N(i)}{N^2} > \epsilon\right) \leqslant \frac{i(N+3-i) - i(N+1-i)}{N^2\,\epsilon^2} = \frac{2i}{N^2\,\epsilon^2} \to 0$$

$$\mathbb{P}\left(\frac{W^N(i) - W^{N-2}(i-1)}{N^2} > \epsilon\right) \leqslant \frac{i(N+1-i) - (i-1)(N-i)}{N^2\,\epsilon^2} = \frac{N}{N^2\,\epsilon^2} \to 0$$

as $N \to \infty$ for all $i \in \{0, 1, \ldots\}$. Therefore, if $\mu^{N_k}\tau_{rN_k}\Theta_{N_k^2}$ converges weakly to μ the same is true for $\mu^{N_k-2}\tau_{rN_k+1}\Theta_{N_k^2}$ and $\mu^{N_k+2}\tau_{rN_k}\Theta_{N_k^2}$. This and (5.9) imply $\tau_1\mu = \mu$ and hence that $\mu \in \mathscr{T}$.

Correlations of the Limit. Since $\mu \in \mathscr{I} \cap \mathscr{T}$, by Proposition 4.11 μ is a convex combination of gamma distributions with mean $r(1-r)$. The covariances of μ equal the variance of the density of μ: for all $i \neq j$,

$$\begin{aligned}&\int \mu(d\eta)\,\eta(i)\,\eta(j) - \left(\int \mu(d\eta)\,\eta(i)\right)^2 \\ &\quad = \iint \nu_\rho(d\eta)\,\eta(i)\,\eta(j)\,\lambda(d\rho) - \left(\int \rho\lambda(d\rho)\right)^2 \\ &\quad = \int \rho^2\lambda(d\rho) - \left(\int \rho\lambda(d\rho)\right)^2 \geqslant 0.\end{aligned} \tag{5.12}$$

Since the joint distribution Q_k of the couple $(\frac{\eta(rN_k+i)}{N_k^2}, \frac{\eta(rN_k+j)}{N_k^2})$ (under μ^{N_k}) converges in distribution to the joint distribution Q of $(\eta(i), \eta(j))$ (under μ), then, as in the discussion following (3.2), by Fatou's Lemma and Skorohod representation for distribution convergence, $\int xy\, dQ \leqslant \liminf_{k\to\infty} \int xy\, Q_k(x, y)$. Since by (5.3) the means converge and $\lim_{k\to\infty} \frac{\sigma^{N_k}(i,j)}{N_k^4} = 0$ (by Theorem 2.5) the covariances of μ are nonpositive.

Conclusion. We have proved that μ is a convex combination of ν_ρ with mean $r(1-r)$ and with zero correlations. This means that $\mu = \nu_{r(1-r)}$. Since each convergent subsequence of $\{\mu^N \tau_{rN} \Theta_{N^2}, N \geqslant 1\}$ converges to $\nu_{r(1-r)}$, so does the sequence. ∎

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